BHARTIYA INSTITUTE OF ENGINEERING & TECHNOLOGY
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श्रेष्ठतमाय कर्मणे

LAB MANUAL

8CE7 Structural Analysis by Matrix Methods Lab

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Subject: Structure analysis by Matrix Method Lab

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Expriment No-1

Object:
flexibility method and the stiffness method.

Theory:-
staticaly indeterminate structure can be analyzed by using the flexibility or the stiffness method.

Flexibility method-
The flexibility method is based upon the solution of equillibrium and Compatibility equations. There will always be as many compatibility eqns. as redundants. It is called the flexibilityes appear in the eqns. of compatibility.
Another name for the method is the force method because forces are the unknowns quantities in eqns. of compatibility.

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Stiffness method-
In the stiffness method displacement (rather than forces) are taken as the unknown quantities. For this reason the method is also called the displacement method. The unknown displacements are obtained by solving equations of equillibrium (rather than equation of compatibility) that contain coefficient in the form of stiffness.

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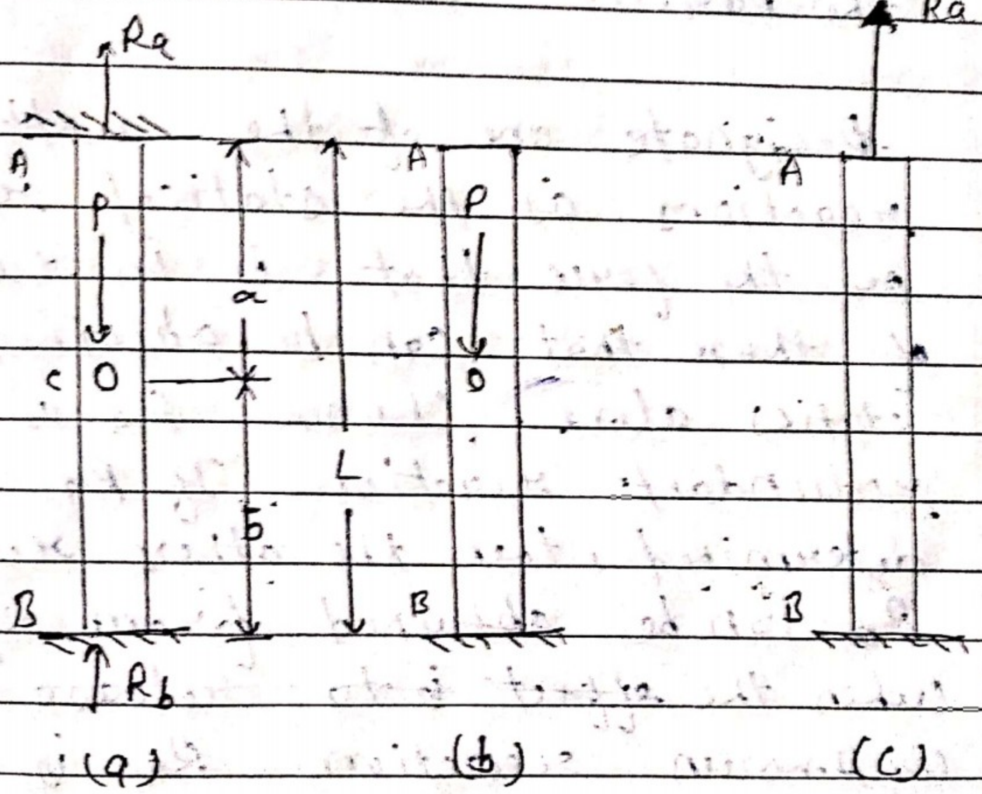
Experiment No-02

Object:

Analysis of a statically indeterminate bar using the flexibility method.

Theory:-



The prismatic bar is attached at both ends to rigid support and is axially loaded by the force P at point C.

from the equation of equillibrium.

$R_a + R_b - P = 0 \longrightarrow (1)$.

A second equation must be obtained from the displacement of the bar.

Designate one of the unknown reactions as the statical redundant, or the force that is in excess of those that can be obtained by statics alone. Choose R_a as the redundant reaction. If R_a can be determined, then the other reaction R_b can be obtained from eq (1). When the ~~effect is to release the~~ unknown reaction R_a is removed from the structure, the effect is to release the support at end A, thereby producing the statically determinate and stable str. shown in fig (b).

Thus from the stand point of having a structure that is capable of supporting load, the reaction of end A is not needed i.e. it is redundant. The str. that remains after releasing the redundent is called the released structure or the primary structure.

Consider the effect of the load P on the displacement of point A in the released structure (fig. (b)). This displacement is,

$$\delta_P = \frac{Pb}{EA} \longrightarrow (2)$$

and is positive downward.
Next, consider the effect of the redundant force R_a on the displacement of point A (fig c).

Note that, although it is an unknown quantity, R_a is now visualized as a load acting on the released structure.

The displacement of point A due to R_a is,

$$\delta_R = \frac{R_a L}{EA} \rightarrow (3)$$

and is positive upward.

The final displacement δ of the point A due to both P and R_a acting simultaneously is found by combine δ_P and δ_R.

Taking downward displacement as positive.

$$\delta = \delta_P - \delta_R$$

Because the actual displacement δ of point A is equal to zero (fig a).

$$\delta = 0 \quad \text{i.e. } \delta_R = \delta_P \rightarrow (4)$$

Substituting from eqn ② and ③ into eqn ④ →

$$\frac{R_a L}{EA} = \frac{P.b}{EA} \Rightarrow R_a = \frac{Pb}{L} \longrightarrow ⑤$$

Thus the redundant reaction has been calculated from and equation related to the displacement of the bar (eqn 4) Knowing the redundant, from eqn ① →

$$R_b = P - R_a = P - \frac{Pb}{L} = \frac{Pa}{L}.$$

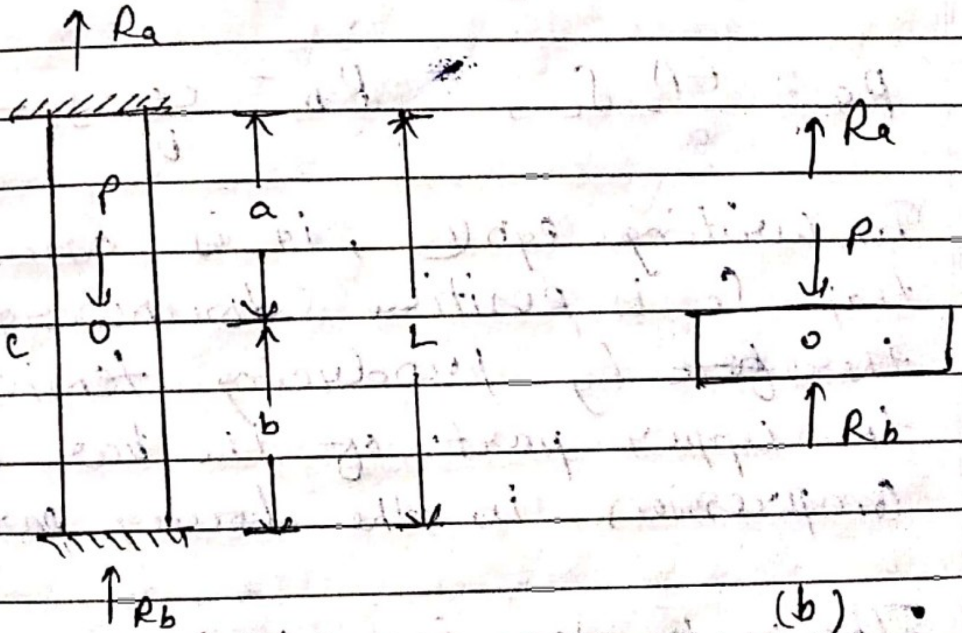
The flexibility method for analysis the statically indeterminate bar may be summarized as follows: first, one of the unknown reactions is selected as the redundant and then released from the structure by cutting through the bar and removing the support.

The released structure, which is statically determinate and stable, is then loaded seperately by the actual load P and by the redundant it self. Next the displacement caused by these two quantity are calculated from the properties of the bar and the material. Then the two displacements are combined into an equation of compatibility (eq 4)

The eq^n of compatibility expresses a condition pertaining to the original structure namely, that the displacement 'δ' at end A is zero. The eq^n of compatibility can be solved for the redundant force R_a finally, the remaining unknown force is found from an equation of equillibrium.

The flexibility method can be used for different types of structures and for str. having many ~~redu~~ redundant forces.

Analysis of a statically indeterminate bar using the stiffness method -



(a) (b)

The ~~ve~~ vertical displacement δ_c of point C, is taken as unknown quantity. The axial forces R_a and R_b in the upper and lower parts of the bar can be expressed in terms of δ_c.

To accomplish this step assume that point C is moved downward by the distance 'δ_c'. Then the upper part of the bar enlongates and lower part shortens by that amount. The axial forces in the two parts are equal to their respective stiffness multiplied by the displacement.

$$R_a = \frac{EA}{a} \delta_c , \quad R_b = \frac{EA}{b} \delta_c \rightarrow (6)$$

In writing eqn (6), it is assumed that δ_c is positive downward, ~~therefore~~ by producing tension in the upper part of the bar and compression in the lower part.

The next step is to isolate point C in the body as a free body (fig b). Acting on the free body are the downward load P, the tensile force R_a in the upper part, and the compressive force R_b in the lower part.

from equillibrium,

$$R_a + R_b - P = 0 \longrightarrow (7)$$

substituting from eqn (6):

$$\frac{EA}{a}\delta_C + \frac{EA}{b}\delta_C = P \longrightarrow (8)$$

which yields,

$$\delta_C = \frac{Pab}{EA(a+b)} = \frac{Pab}{EAL} \longrightarrow (9)$$

Hence, from eqn (6)

$$R_a = \frac{Pb}{L}, \quad R_b = \frac{Pa}{L} \longrightarrow (10)$$

The stiffness method for analyzing the statically indeterminate bar may be summarized as follows.

First, select a suitable displacement as the unknown quantity. A displacement will be suitable if the force in the individual parts of the structure can be expressed in terms of that displacement.

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Next, the force are related by an eqn of equilibrium (eqn 7). Then the expressions giving the forces in terms of the unknown displacements are substituted into the eqn of equilibrium, thereby producing an eqn with only the selected displacement as an unknown (eqn 8). Note the coefficients of δ_c in this eqn are the stiffnesses. This eqn is solved for the unknown displacement (eq. 9) and finally, the forces are found from the displacement (eqn 10).

When comparing the flexibility and stiffness method, it is seen that the flexibility method requires the soln of eqn of compatibility for ~~the~~ unknown forces whereas the stiffness method requires the solution of equilibrium for unknown displacement.

Experiment No. 3

Object:
A simply supported beam resisting Carrying end-moments by matrix method

Theory:
Consider a SSB resisting moments M_1 and M_2 applied at its ends,

The flexibility relates the end rotations $\{\theta_1, \theta_2\}$ to the end moment $\{M_1, M_2\}$

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

The flexibility coefficients F_{ij} may be obtained from castigliano's 2nd theorem, $\theta_i = \partial U^* (M_i) / \partial M_i$

The applied moments M_1 and M_2 are in equillibrium with the reactions forces V_1 and V_2,

$V_1 = (M_1 + M_2)/L$ and

$V_2 = -(M_1 + M_2)/L$

$$V(x) = \frac{M_1 + M_2}{L}$$

$$M(x) = M_1\left(\frac{x}{L} - 1\right) + M_2\frac{x}{L}$$

The total potential energy of beam with these forces and moments is:

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + \frac{1}{2}\int_0^L \frac{V^2}{G(A/\alpha)}\,dx$$

By castigliano's theorem,

$$\theta_1 = \frac{\partial U}{\partial M_1}$$

$$= \int_0^L \frac{M(x)\frac{\partial M(x)}{\partial M_1}}{EI}\,dx + \int_0^L \frac{V(x)\frac{\partial V(x)}{\partial M_1}}{G(A/\alpha)}\,dx$$

$$= \left(\int_0^L \frac{\left(\frac{x}{L} - 1\right)^2}{EI}\,dx + \int_0^L \frac{\alpha\,dx}{GAL^2}\right)M_1$$

$$+ \left(\int_0^L \frac{x/L\,(x/L - 1)}{EI}\,dx + \int_0^L \frac{\alpha\,dx}{GAL^2}\right)M_2$$

and: $Q_2 = \dfrac{\partial u}{\partial M_2}$

$$= \int_0^L \frac{M(x)\dfrac{\partial M(x)}{\partial M_2}}{EI}\,dx + \int_0^L \frac{V(x)\dfrac{\partial V(x)}{\partial M_2}}{G(A/\alpha)}\,dx$$

$$= \left(\int_0^L \frac{x/L\,(x/L-1)}{EI}\,dx + \int_0^L \frac{\alpha\,dx}{GAL^2}\right)M_1 +$$

$$\left(\int_0^L \frac{(x/L)^2}{EI}\,dx + \int_0^L \frac{\alpha\,dx}{GAL^2}\right)M_2$$

Or in matrix form,

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{bmatrix} \left(\displaystyle\int_0^L \frac{(x/L-1)^2\,dx}{EI} + \int_0^L \frac{\alpha\,dx}{GAL^2}\right) & \left(\displaystyle\int_0^L \frac{x/L\,(x/L-1)\,dx}{EI} + \int_0^L \frac{\alpha\,dx}{GAL^2}\right) \\ \left(\displaystyle\int_0^L \frac{x/L\,(x/L-1)\,dx}{EI} + \int_0^L \frac{\alpha\,dx}{GAL^2}\right) & \left(\displaystyle\int_0^L \frac{(x/L)^2\,dx}{EI} + \int_0^L \frac{\alpha\,dx}{GAL^2}\right) \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

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Beam element stiffness matrices:

for prismatic beams E, A and I are constant along the length and the flexibility relationship is,

$$\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} \frac{L}{3EI} + \frac{1}{G(A/k)L} & -\frac{L}{6EI} + \frac{1}{G(A/k)^2} \\ -\frac{L}{6EI} + \frac{1}{G(A/k)^2 L} & \frac{L}{3EI} + \frac{1}{G(A/k)L} \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$

To neglect shear deformation, set $\alpha = 0$

The stiffness relationship is the inverse of the flexibility relationship, and for prismatic members;

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{(4+\phi)EI}{(1+\phi)L} & \frac{(2-\phi)EI}{(1+\phi)L} \\ \frac{(2-\phi)EI}{(1+\phi)} & \frac{(4+\phi)EI}{(1+\phi)L} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

where,

$$\phi = \frac{12EI}{G\left(\frac{A}{\alpha}\right)L^2} = 24\alpha(1+\nu)\left(\frac{r}{L}\right)^2$$

and r is the "radius of gyration" of cross section $r = \sqrt{\frac{I}{A}}$

To neglect the shear deformation, set $\phi = 0$

$$\begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

Beam element stiffness matrix in local coordinates, K:-

The beam element stiffness matrix 'K' relates the shear force and bending moment at the end of the beam $\{V_1, M_1, V_2, M_2\}$ to the deflection and rotation at the end of the beam $\{\Delta_1, \theta_1, \Delta_2, \theta_2\}$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \theta_1 \\ \Delta_2 \\ \theta_2 \end{Bmatrix}$$

The elements of this four-by-four stiffness matrix may be derived from eqn ① using arguments of equilibrium and symmetry.

The second column of the stiffness matrix is the set of forces and moments corresponding to the following set of displacements and rotations.

$$[\Delta_1 = 0 \ , \ \theta_1 = 1 \ , \ \Delta_2 = 0 \ , \ \theta_2 = 0]$$

from eqn ①, we know,

$$M_1 = \frac{4EI}{L} = K_{22}$$

$$M_2 = \frac{2EI}{L} = K_{12}$$

from equilibrium we know,

$$V_1 = (M_1 + M_2)/L = 6EI/L^2 = K_{12}$$

$$V_2 = -(M_1 + M_2)/L = -6EI/L^2 = K_{32}$$

This completes the second column of K. Similarly the fourth column of K is the set of forces and moments corresponding to,

$$\{ \Delta_1 = 0,\ \theta_1 = 0,\ \Delta_2 = 0,\ \theta_2 = 1 \}$$

These forces and moments are

$$M_1 = 2EI/L = K_{24}$$
$$M_2 = 4EI/L = K_{44}$$

$$V_1 = (M_1 + M_2)/L = 6EI/L^2 = K_{14}$$
$$V_2 = -(M_1 + M_2)/L = -6EI/L^2 = K_{34}$$

Now the first column of K is the set of forces and moments ~~to~~ corresponding to.

$$\{ \Delta_1 = 1,\ \theta_1 = 0,\ \Delta_2 = 0,\ \theta_2 = 0 \}$$

from argument of symmetry (of the element stiffness matrix)
We, know,

$$M_1 = K_{21} = K_{12} = \frac{6EI}{L^2}$$

and,

$$M_2 = K_{41} = K_{14} = \frac{6EI}{L^2}$$

And from equilibrium,

$$V_1 = (M_1 + M_2)/L = 12EI/L^3 = K_{11}$$

and

$$V_2 = -(M_1 + M_2)/L = -12EI/L^3 = K_{31}$$

Finally, the third column of K is the set of forces and moments corresponding to,

$$\{\Delta_1 = 0,\ \theta_1 = 0,\ \Delta_2 = 1,\ \theta_2 = 0\}$$

from argument of symmetry (of the element stiffness matrix)

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We know,

$$M_1 = K_{23} = K_{32} = -6EI/L^2$$

$$M_2 = K_{43} = K_{34} = -6EI/L^2$$

And from equilibrium,

$$V_1 = (M_1 + M_2)/L = 12EI/L^3 = K_{13}$$

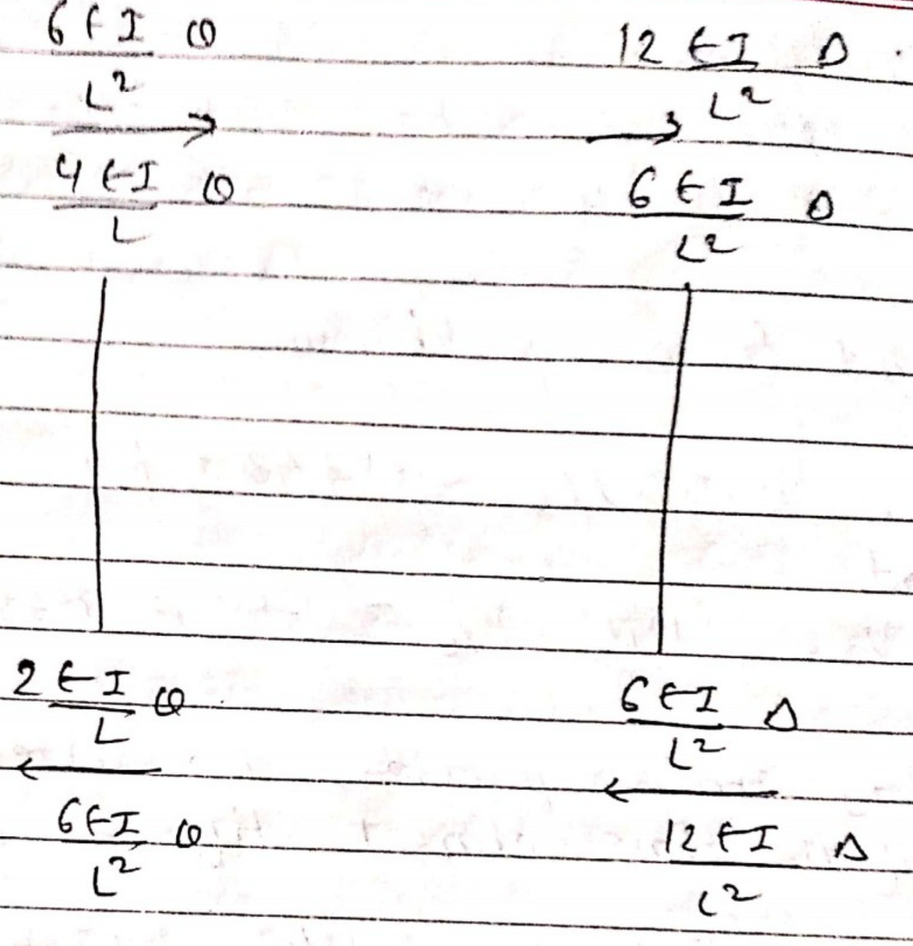
and

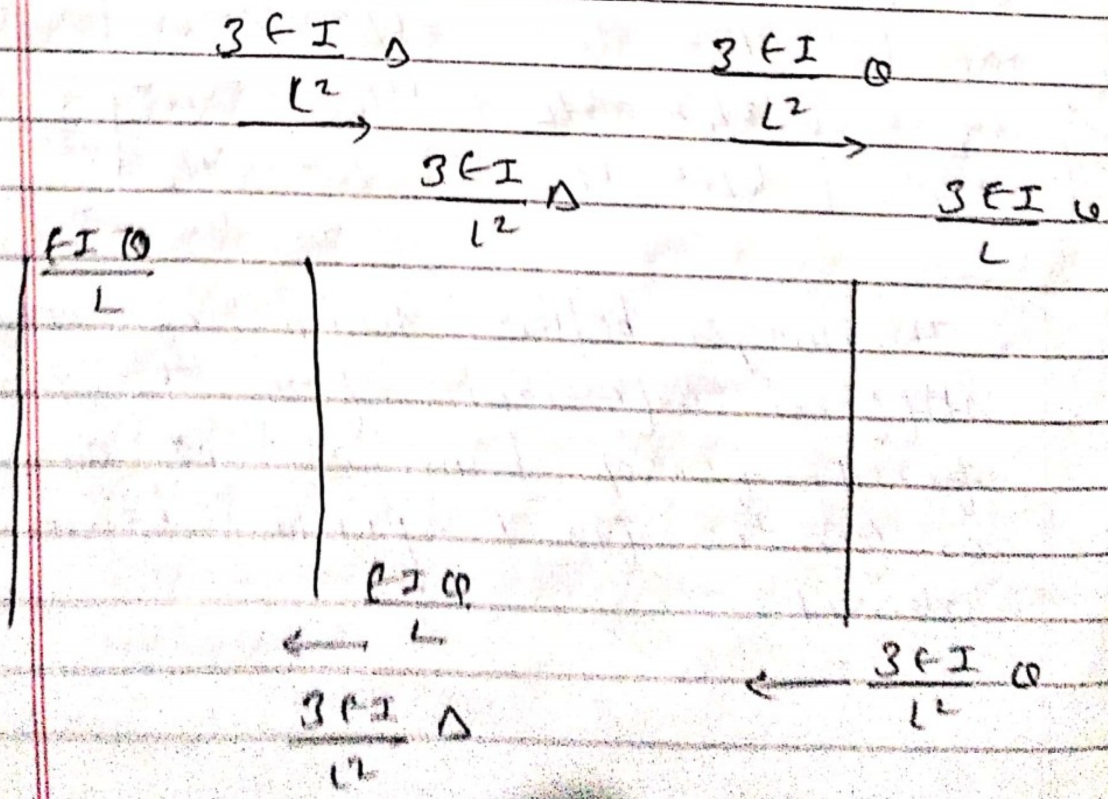
$$V_2 = -(M_1 + M_2)/L = 12EI/L^3 = K_{33}$$

This analysis provides the sixteen terms of the beam element stiffness matrix.

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \theta_1 \\ \Delta_2 \\ \theta_2 \end{Bmatrix}$$

The images below summerize the stiffness coefficients for the standard fixed-fixed beam element as well as for the fixed-pinned beam element.


$\frac{6EI}{L^2}\theta$
$\frac{12EI}{L^2}\Delta$
$\frac{4EI}{L}\theta$
$\frac{6EI}{L^2}\theta$
$\frac{2EI}{L}\theta$
$\frac{6EI}{L^2}\Delta$
$\frac{6EI}{L^2}\theta$
$\frac{12EI}{L^2}\Delta$


$\frac{3EI}{L^2}\Delta$
$\frac{3EI}{L^2}\theta$
$\frac{3EI}{L^2}\Delta$
$\frac{3EI}{L}\theta$
$\frac{EI}{L}\theta$
$\frac{EI}{L}\theta$
$\frac{3EI}{L^2}\theta$
$\frac{3EI}{L^2}\Delta$

Experiment NO-04

Object:
Introduction about finite element Method (FEM).

Theory:-

The finite element method (FEM) is a numerical technique for solving problems which are described ~~is~~ by partial differential equations or can be formulated as functional minimization. A domain of interest is represented as an assembly of finite elements.

Two features of the FEM are worthy to be mentioned-
→ Piece-wise approximation of physical field on finite elements provides good precision even with simply approximating functions (increasing the number of elements we can acheive any precision)

→ Locality of approximating leads to sparse equation systems for a discretized problem. This helps to solve problems with very number of nodal unknowns.

Procedure -
There are following ~~foll~~ steps are of the finite element solution procedure :-

1. Discretize the continuum.
The first step is to divide a soln region into finite elements. The finite element mesh is typically generated by a preprocessor program. The description of mesh consits of several arrays main of which are nodal coordinations and element connectivities.

2. Select interpolation functions.
Interpolation functions are used to interpolate the field variables over

the element. often, polynomials are selected as interpolation functions. The degree of the polynomial depends on the number of nodes assigned to the element.

3. find the element properties.
The matrix equation for the finite element should be established which relates the nodal values of the unknown function to other parameter. for this task different approaches can be used; the most convenient are: the variational approach and the Galerkin method.

4. Assemble the element equations.
To find the global equation system for the whole solution region we must assemble all the element equations. Element connectivities are used for the assembly process. Before solution, boundary conditions (which is not accounted in element equations) should be imposed.

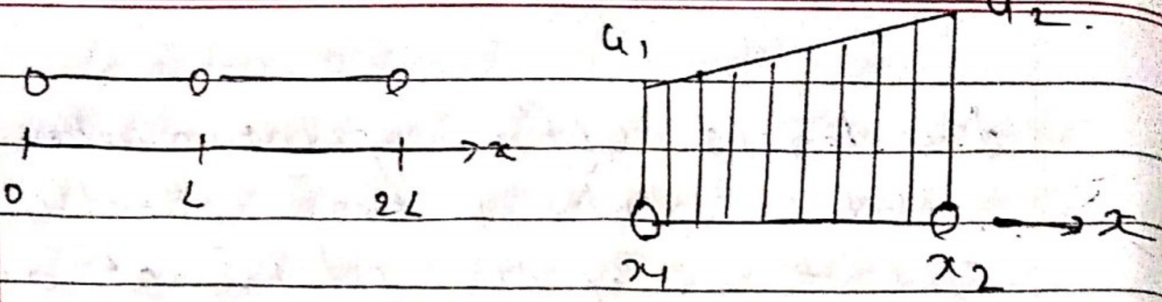


fig: Two 1-D linear elements and function interpolation inside element

5. Solve the global equation system.
The finite element global eqn system is typically sparse, symmetric and positive define. Direct and iterative method can be used for solution. The nodal values of the sought function are produced as a result of the solution.

6. Computer additional results.
In many cases we need to calculate additional parameters, for example, in mechanical problems strains and stresses are of interest in addition to displacements, which are obtained after solution of the global eqn system.

⇒ Method of finite element equations

1) Galerkin method:

Let us use simple ~~one-D~~ 1-D example for the explanation of finite element formulation using the Galerkin method. Suppose that we need to solve numerically the following differential equation.

$$\cancel{a\frac{d^2}{d}} \quad a\frac{d^2u}{dx^2} + b = 0 \qquad 0 \le x \le 2L \qquad \text{(1)}$$

with boundary conditions

$$u|_{x=0} = 0$$

$$a\frac{du}{dx}\Big|_{x=2L} = R \qquad \text{(2)}$$

where u is an unknown solution. We are going to solve the problem using two linear 1-D finite elements as shown in fig.

Consider a finite element presented on the right of figure. The element has two nodes and approximation of the function $u(x)$ can be done as follows

$$u = N_1 u_1 + N_2 u_2 = [N]\{u\} \longrightarrow (1)$$

$$[N] = [N_1 \quad N_2]$$
$$\{u\} = \{u_1 \quad u_2\}$$

where $N_1 = 1 - \dfrac{x - x_1}{x_2 - x_1}$

$$N_2 = \frac{x - x_1}{x_2 - x_1} \longrightarrow (4)$$

N_i are the so called shape functions

2) Variational ~~method~~ formulation:
The differential eqn

$$a\frac{d^2u}{dx^2} + b = 0 \quad , \quad 0 \le x \le 2l$$

$$u\Big|_{x=0} = 0$$

$$a\frac{du}{dx}\Big|_{x=2l} = R$$

with $\alpha = EA$ has the following physical meaning in solid mechanics. It described tension of the 1-D bar with cross-sectional area A made of material with the elasticity modulus E and subjected to a distributed load b and a concentrated load R at its right end as in fig below-
Such problem can be formulated in terms of minimizing the potential energy functional Π.

$$\Pi = \int_L \frac{1}{2} a \left(\frac{du}{dx}\right)^2 - \int_L b u \, dx - R u\Big|_{x=2L}$$

$$u\Big|_{x=0} = 0$$

formulation of finite element equations-

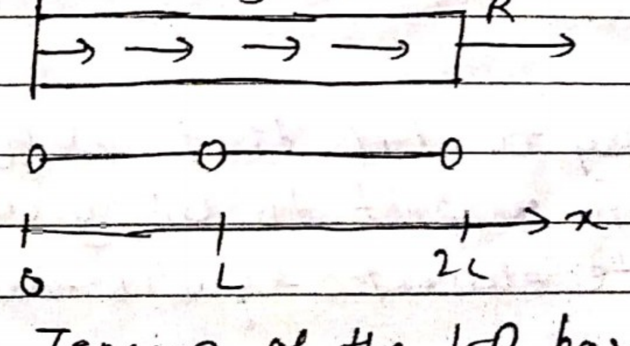


fig: Tension of the 1-D bare subjected to a distributed load and a point load

Using representation of $\{u\}$ with shape function (3-4) we can write the value of potential energy for the second finite element as:

$$\Pi_e = \int_{x_1}^{x_2} \frac{1}{2} a \{u\}^T \left[\frac{dN}{dx}\right]^T \left[\frac{dN}{dx}\right] \{u\} dx - \int_{x_1}^{x_2} \{u\}^T [N]^T b\, dx - \{u\}^T \begin{Bmatrix} 0 \\ R \end{Bmatrix}$$

The condition for the min. of Π is

$$\delta \Pi = \frac{\partial \Pi}{\partial u_1} \delta u_1 + \dots + \frac{\partial \Pi}{\partial u_n} \delta u_n = 0$$

which is equivalent to

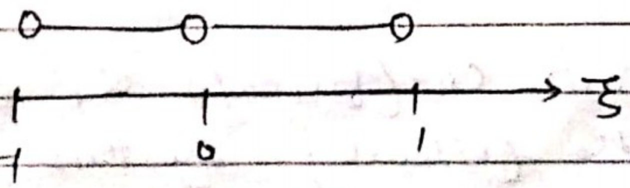
$$\frac{\partial \Pi}{\partial u_i} = 0 \quad , \quad i = 1 \dots n$$

It is easy to check that differentiation of Π in respect to u_i gives the finite element equilibrium eqn

$$\int_{x_1}^{x_2}\left[\frac{dN}{dx}\right]^T EA\left[\frac{dN}{dx}\right]dx\,\{u\} - \int_{x_1}^{x_2}[N]^T b\,dx - \begin{Bmatrix}0\\R\end{Bmatrix} = 0$$

Example :
~~with shape functions, any field inside element is presented as;~~
$\cancel{u(\xi) = \sum N_i u_i \; , \; i = 1,2,3}$

Example: obtain shape functions for the 1-D quadratic element with three nodes. Use local coordinate system $-1 < \xi \leq 1$.



Sol^n- With shape functions, any field element is presented as,
$$u(\xi) = \sum N_i u_i \; , \; \forall \; i = 1,2,3$$

At nodes the approximated functions should be equal to its nodal value.

$u(-1) = u_1$
$u(0) = u_2$
$u(1) = u_3$

Since the element has three ~~poly~~ nodes the shape functions can be quadratic polynomials (with three coefficients): The shape function N_1 can be written as:

$N_1 = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2$

Unknown coefficients α_i are defined from the following system of equations:

$N_1(-1) = \alpha_1 - \alpha_2 + \alpha_3 = 1$
$N_1(0) = \alpha_1 = 0$
$N_1(1) = \alpha_1 + \alpha_2 + \alpha_3 = 0$

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The solution is: $\alpha_1 = 0$, $\alpha_2 = -1/2$, $\alpha_3 = 1/2$.

Thus, the shape function N_1 is equal to:

$$N_1 = \frac{-1}{2}\xi(1-\xi)$$

Similarly it is possible to obtain that the shape functions N_2 and N_3 are equal to:

$$N_2 = 1-\xi^2$$
$$N_3 = \frac{1}{2}\xi(1+\xi)$$